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WHAT A TEACHER SAYS AND WHAT A STUDENT UNDERSTANDS

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Abstract

Based on the view that the difficulties students experience in learning mathematics are in part due to a lack of a shared precise language, and the recent emphasis on the importance of language in the development of mathematical proficiency through newly formed reform practices around the world, we analyzed student work to determine what language was used that potentially led to common misconceptions. The sample consisted of 69 first-semester freshmen at a University in the Midwest United States. Data were analyzed qualitatively, with the three authors discussing student work and language use until 100% agreement was found. Focusing on rational expressions, polynomials, and number systems; we found major misconceptions related to negative sign, simplifying rational expressions, multiplying binomials, and conceptualizing fractions. We analyzed students' explanations and reasoning to determine the potential misuse of language for each misconception. In this paper, we illustrate examples of misconceptions, present evidence for imprecise and incoherent language use that potentially supported the development of each misconception, and recommend coherent and precise language in all grades, K-12, to develop both procedural and conceptual understanding, which will be beneficial to students as they advance from high school into college.

Keywords: *Algebra, misconceptions, mathematical language, mathematics learning.*

1. Introduction

High school students in the United States recently overwhelmingly scored below college-readiness levels in mathematics, according to the National Assessment of Educational Progress (NAEP,2019). Results showed that less than 25% of students performed proficiently, even though 77% of the lowest performing students reported taking advanced mathematics courses in high school. This leads to higher levels of students taking developmental mathematics courses in two-year colleges (59%) and four-year colleges (33%), with an average student taking two to three successive courses. At least half of those students do not complete all of their required developmental math courses, and only about 20% of students who complete all of their developmental math courses successfully complete college level math courses (Brock et al., 2016). These struggles with mathematical success have been attributed to the procedural nature of mathematics instruction in high schools (Zenati, 2019) rather than a conceptual development.

Difficulties students experience in learning mathematics concepts are in part due to a lack of a shared precise language, despite the recent emphasis on the importance of language in the development of mathematical proficiency in newly formed reform practices. In fact, mathematical discourse was identified as one of six salient practices present in reform mathematics studies from around the world in the past decade (Morrissey et al., 2022). In the United States, reform math is encapsulated by the Common Core State Standards for Mathematics (CCSSM, 2010), which advocates for precise mathematical language as one of the core standards.

Given the importance of precise language use, coupled with the high percentage of college students who are not mathematically college-ready, Popovic et al. (2022) recommended investigating "how treatment of a concept in a restricted context in high school CCSSM-aligned textbook may lead to a potential conflict and difficulty of concept development in the broader context, later in the college" (p. 863). This recommendation led us to analyze student work to determine what language was used that potentially led to common misconceptions.

2. Literature review

The types of misconceptions that students encounter in mathematics, particularly in Algebra, have long been investigated by many researchers (e.g., An, 2004; Barcellos, 2005; Fuchs & Menil, 2009; Jurkovic, 2001, Holmes et al., 2013; Powell & Nelson 2020), given the importance of reviewing student errors and misconceptions for an improved student conceptual understanding (Durkin 2009; Grobe & Renkle, 2007; NCTM, 2007). In addition to identifying the types of misconceptions that arise across grade levels and mathematical content areas, some studies explored the underlying causes for these misconceptions. Researchers reported that misconceptions arise as a result of the imprecise, inaccurate, and inconsistent use of mathematical language (e.g., Baidoo, 2019; Jaffar & Dindyal, 2011; Lager, 2006; Popovic et al., 2022; Powell & Nelson, 2020). In other words, imprecise mathematical language leads to ambiguity which hinders students' understanding and concept development (Jaffar & Dindyal, 2011; Lager, 2006).

Rational number concepts and algebraic fractions have been one of the topics in which misconceptions arise across all levels, i.e., elementary, secondary, and post-secondary (Barcellos, 2005; Baidoo, 2019; DeWolf et al., 2016; Gabriel et al., 2013; McAllister & Beaver, 2012; Powell & Nelson, 2020; Vamvakoussi & Vosniadou, 2004). For example, understanding that there are infinitely many numbers between any two real numbers is difficult for students to comprehend (Vamvakoussi & Vosniadou, 2004); or distribution of minus sign over rational expressions and invalid form of cancellation in rational expressions are problematic for students (Barcellos, 2005). In this respect, researchers suggested that educators at all levels (i.e., elementary, secondary, postsecondary) should provide precise and consistent mathematical language in the rational number concepts and algebraic fractions. We focused on undergraduate students because previous research indicates many postsecondary students continue to struggle with rational number concepts. This previous research (e.g., DeWolf et al., 2016; Gabriel et al., 2013; McAllister & Beaver, 2012) identified error patterns, and reported diversity of errors and the persistent difficulties with rational numbers and expressions that university students demonstrate. Differently from the previous research, in addition to identifying the error patterns, we investigated the underlying imprecise language that potentially led to such misconceptions and errors.

3. Methodology

Sample consisted of first-semester freshmen students at a University in the Midwest United States. The students were enrolled in the remedial mathematics course, Preparation for Calculus, based on their mathematics placement exam score. During the first class, students were asked to complete the pre-assessment test consisting of 14 items. Each problem asked students to circle one of the answer choices and to explain their reasoning. The answer choices were based on the common errors students make in algebra. Students were given 45 minutes to complete the test. The instructor ensured students that the purpose of the test was to determine their current knowledge with respect to the concepts assessed. To motivate students to do their best to answer each question, the instructor informed students that one point would be awarded for each problem for which a student circled the answer and provided an explanation, regardless of the correctness of the answer. Hence, the students could earn 14 points that would be counted as extra credit points in the exams category. A total of 69 student work was collected.

Data were analyzed qualitatively. The focus was on identifying language students use to explain their reasoning, and determining how the language used might lead to student inappropriate understanding of fractions, rational expressions and the negative sign. The three authors discussed a selected sample of ten students' work until they reached a 100% agreement in interpreting student reasoning based on their explanations.

4. Results

Figure 1 includes the problems posed to the students. Questions 1, 5, 9, and 10 assessed students' understanding of fractions and rational expressions, while questions 2, 7, and 10 assessed students' understanding of the negative sign.

Figure 1. Pre-Assessment Problems 1, 2, 5, 7, 9 and 10.

Question 1: Which of the equalities below you think is true for all real values of $a \neq 0, b \neq 0, x$ and y? (Circle **one** equality)

$$\frac{x}{a}+\frac{y}{b}=\frac{x+y}{a+b} \qquad \frac{x}{a}+\frac{y}{b}=\frac{xb+ya}{ab} \qquad \frac{x}{a}+\frac{y}{b}=\frac{x+y}{ab}$$

Explain your choice.

Question 2: Which of the equalities below you think is true for all real values of $a, b,$ and c? (Circle **one** equality)

$$a-(c-b)=a-c-b \qquad a-(c-b)=a-c+b$$

Explain your choice.

Question 5: Which of the equalities below you think is true for all real values of $a, b,$ and c? (Circle **one** equality)

$$\frac{b+ax}{a}=b+x \qquad \frac{b+ax}{a}=\frac{b}{a}+x$$

Explain your choice.

Question 7: Circle the numbers below that you think can be assigned to $-b$, where b is a real number?

$$1 \quad -1 \quad 2 \quad -2 \quad \frac{2}{3} \quad -\frac{2}{3} \quad 2.66 \quad -2.66 \quad \pi \quad -\pi$$

Explain your choice.

Question 9: Do you think that $\frac{x(x-1)(x-2)}{x-1}=x(x-2)$ true for all real values of x? (Circle **one** answer)

YES NO

Explain your choice.

Question 10: Which of the equalities below you think is true for all real values of $a \neq 0, b \neq 0, x$ and y? (You may circle more than one answer)

$$-\frac{x+5}{x-1}=\frac{-x-5}{x-1} \qquad -\frac{x+5}{x-1}=\frac{-x-5}{-x+1} \qquad -\frac{x+5}{x-1}=\frac{x+5}{-x+1}$$

Explain your choice.

Table 1 shows the number and percentages of the correct and incorrect answers for each question. Two students have not provided answers to each of the questions 7 and 10.

Table 1. Pre-Assessment Problems 1, 2, 5, 7, 9 and 10.

Question	1	2	5	7	9	10
Correct	47 (68%)	60 (87%)	33 (48%)	10 (14%)	18 (26%)	19 (28%)
Incorrect	22 (32%)	9 (13%)	36 (52%)	57 (83%)	51 (74%)	48 (71%)

4.1. Misconceptions about fractions

Forty-seven (68%) of the students responded correctly to the first question. However, a qualitative analysis of their explanations showed that only five students justified their answer choice by referring to the least common denominator and equivalent fractions. A half of the students stated that they achieved the correct answer because they “do the same to the numerator and the denominator,” “multiply top and bottom number by the other fractions’ denominator” or “multiply fraction x/a by *b* and fraction y/b by *a*.” One-third of the students explained their correct answer choices by “do the cross multiplication.” Finally, a few students substituted specific values for *x*, *y*, *a*, and *b* to check which of the equalities is true, and one student guessed. Students who selected the first answer explained that they “add across the top and the bottom numbers” or “add across the numerator and the denominator number.” Finally, the students who selected the third answer explained that you need a common denominator in order to add the numerators. Students who answered Question 5 correctly (33 or 48%) explained their answer choice in one of the following ways; “you have to divide each individual value by *a*” or “if I multiply both sides by *a* you get the same answer b+ax.” All students who answered the question incorrectly said that “you can cancel the same value in the numerator and the denominator.” A few students based their answers, whether correct or incorrect (claimed both are true equalities), by substituting one combination of specific values for *a*, *b*, and *x*. Only 18 (26%) of the students correctly answered Question 9, explaining that the two expressions are not equal if x=1, because the expression on the right side of the equal sign would be undefined. Remaining 51 (74%) students believed that the two expressions are equal because you cancel x-1 in the numerator and the denominator. For Question 10, 31 (45%) students selected the second answer, explaining that "you distribute a negative to each number (value) in the fraction," 11 (16%) students believed you can "distribute a negative to the numerator only," while three students stated that all three answers are practically the same. The students who chose both the first and the third explained that "dividing one negative numerator or denominator gives a negative value" or "you can apply the negative to either the denominator or the numerator" and similar. Only one student explained that "the negative sign in front of the fraction means the same as multiplying by a -1, which is -1/1 or 1/(-1)."

4.2. Misconceptions about the negative sign

As expected, the majority of the students responded correctly to Question 2. However, their explanations talked about "distributing a negative to the variables in the parentheses," or that "a negative turns *b* into a positive." The students who selected an incorrect answer said that "in addition and subtraction the parentheses don't matter." Only 10 (14%) students answered Question 9 correctly and circled all listed numbers, stating that "all numbers are real." Students who did not think all listed numbers could be assigned to -b, and explained "only positive numbers can be used because a negative will turn it into a negative number" or "only negative numbers can be used because the "negative" says that *b* is a negative number." Other students showed a lack of understanding of real numbers, by selecting the integers only, the integers and decimals, and all numbers except and -π.

5. Discussion

Similar to the previous literature, we found misconceptions in distribution of minus sign over rational expressions and invalid form of cancellation in rational expressions, as well as an incomplete understanding of fractions which resulted in mistreatment of algebraic fractions. Students' written explanations allowed us to identify some of the potential inconsistent language that might have led to such difficulties.

The qualitative analysis of students' answers to questions 1, 5, 9, and 10 revealed that the majority of the students conceptualize a fraction as *two numbers separated by a fraction bar*, rather than as a number representing a single quantity. Students' use of "top" and "bottom" language in their explanations clearly evidenced their discrete understanding of fractions. Thus, we hypothesize that this may be due to the language commonly used when working with fractions, "*a* over *b*" rather than "*a* divided by *b;*" "top" and "bottom" numbers instead of the "numerator" and the "denominator." While many students were able to select a correct answer, their explanations demonstrated major misconceptions about mathematical concepts other than fractions. For example, students used phrase *cross multiplication* to explain addition of fractions, rather than creating equivalent fractions to the given fractions that have the common denominator. This may be related to learning the "butterfly method" of cross-multiplying fractions to determine which fraction is greater, which is itself problematic (Karp et al., 2015). The main reason for the error students make in question 5 might be that the teachers say "we cancel the common (same) number (value) in the numerator and the denominator" instead of "we divide the numerator and the denominator by the common (same) factor."

Students' errors in problem 2 and 10 may be caused by talking about "distributing a negative (sign)" rather than distributing a -1 or taking the opposite of the binomial c-b. This combination of vocabulary, "distributing" and "negative," is a further example of students' memorizing content but not having the conceptual understanding to apply it appropriately. As shown in answers in problem 7, the majority of the students believed that the negative sign in expression -b either denotes that "*b* is a negative number" and thus selected negative numbers or "-b has to be a negative number" and thus selected positive numbers as appropriate. Additionally, students demonstrated a lack of understanding of the real numbers. For example, a student selected all numbers except and -π, explaining that "real numbers encompass numbers that can be defined. Pi goes until infinity so probably not" or choose all positive numbers "because real numbers cannot be negative."

It is interesting that some of the students referred to the expressions on each side of the equal sign as *equations*. A student explained the correct answer choice by "the second equation is true as they are both the same equation after simplification," while another student explains the incorrect answer choice by "the variable *a* cancels out so equation is the same as b+x." This might be related to the common understanding of the equal sign as "do this," which is mostly communicated in early grades, rather than "equality of the quantities on both sides of the equal sign," which is necessary for learning concepts in advanced mathematics courses.

Clearly, the lack of a shared precise language fosters misconceptions, based on a lack of conceptual understanding. As Karp et al. (2015) recommended, intentional, consistent, and precise use of rules, terminology, and notation are needed in order for students to focus on new, complex mathematical ideas.

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